

Complementary relation of quantum coherence and quantum correlations in multiple measurements

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Quantum coherence and quantum correlations lies in the central of quantum mechanics and both are considered as fundamental reasons for significance deviation of quantum mechanics from classical mechanics which are often valued as precious resources in quantum information processing tasks. We present a group of complementary relations for quantum coherence and quantum correlations specifically thermal discord and conditional information in scenarios of multiple measurements experiments which would deepen our understanding of quantum coherence and quantum correlations.

Introduction.— Quantum coherence namely principle of superposition of quantum states is one of the cornerstones of quantum theory. It is widely believed to provide advantages in quantum information and computation tasks over classical methods. The rigorous framework of coherence quantification is only recently been introduced in which several criteria were proposed for any measure of coherence to satisfy [1–10]. The quantification framework of quantum coherence makes it possible for quantitative studies which draws intensive attention [11–19].

Coherence is also closely related to quantum correlations such as entanglement and quantum discord [20–23]. Thermal discord [24–26] and conditional information [27] are two well known kinds of quantum correlation. Thermal discord also known as one-way deficit is often used in the study of quantum thermodynamics such as the study of Maxwell’s Demon [25, 26]. While, conditional information plays an important role in studying quantum entanglement [28–30] and state merging [31, 32]. A negative value of conditional information signals existence of entanglement.

Apart from these quantuness namely quantum coherence and quantum correlations such as thermal discord and conditional information, Heisenberg uncertainty principle [33–40] is also of intensive study in quantum information. These studies of uncertainty relation especially that of entropic complementary relations [35–40] not only provide deeper understanding of the foundation of quantum mechanics but also give useful tools for researches in quantum information. We will use these uncertainty relations especially those of Ref. [38, 39] to give a group of complementary relations for quantum coherence, thermal discord and conditional information. These results will benefit us in deeper understanding of quantum coherence, thermal discord and conditional information. Specically, we given complementary constraints on quantum coherence and basis-dependent thermal discord with respect to multiple measurement basis. We examine

coherence in a one-particle system and thermal discord in a bipartite system by implementing all the measurements on a single subsystem. We also present a complementary relation on post-measurement conditional information in multipartite system under multi measurements. The conditional information will be considered between a chosen subsystem with rest of the subsystems and measurements would also be applied to that specifically chosen subsystem.

Coherence quantification and the definitions of thermal discord and conditional information. A natural measure of quantum coherence is defined as a pseudo-distance formulated by relative entropy between the studied quantum state with the nearest incoherent state which can be proven to be a diagonal density matrix with diagonal elements equal to the original density matrix [1]

$$C_{\text{RE}}^{(\mathcal{M})}(\rho) = S(\tilde{\rho}^{(\mathcal{M})}) - S(\rho). \quad (1)$$

Here, $S(\cdot)$ is the well-known von Neumann entropy, $\mathcal{M} := \{\Pi_i := |i\rangle\langle i|\}$ is a projective measure, and $\tilde{\rho}^{(\mathcal{M})} := \sum_i \Pi_i \rho \Pi_i$ the post-measurement state. Because coherence measures such as $C_{\text{RE}}^{(\mathcal{M})}(\cdot)$ are basis dependent. Therefore finding the relation between coherence of the same state with respect to different measurement basis \mathcal{M} would be very interesting and valuable for deeper understanding of coherence.

Thermal discord was defined as [24, 26]

$$D_{\text{th}}^{(k)}(\text{B}|\text{A}) := \sum_{i=1}^d p_i^{(k)} S(\tilde{\rho}_{\text{B}i}^{(k)}) + S(\tilde{\rho}_{\text{A}}^{(k)}) - S(\rho_{\text{AB}}). \quad (2)$$

$p_i^{(k)}$ is probability of obtaining result i in measurement $\mathcal{M}_{\text{A}}^{(k)}$ upon subsystem A, $\tilde{\rho}_{\text{B}i}^{(k)}$ is the corresponding post-measure state of subsystem B, and $\tilde{\rho}_{\text{A}}^{(k)}$ is the state of subsystem A after that measurement without knowing outcome. As one can see, this definition is also measurement-dependent. We can actually make a

measurement-independent definition by requiring a minimization over all projective measurements. But this would complicate the calculation and make it difficult to obtain a closed expression. Also, in the lab we would always choose specific measurements when the optimal measurement is not available. In the meantime to gain more information, one would implement more than one measurement. In such a situation, it would be very helpful if we know any relation between the thermal discord corresponding to different measurements.

On the other hand, conditional information on joint system AB is defined as $S(A|B) := S(\rho_{AB}) - S(\rho_B)$ [27]. We denote conditional entropy after measurement \mathcal{M}_A on system A as $S(\mathcal{M}_A|B) = S(\tilde{\rho}_{AB}) - S(\tilde{\rho}_B)$. We find it is interesting to have a complementary relation on post-measurement conditional information in a multipartite system.

Complementary relation of quantum coherence.— It has been shown that the entropic uncertainty relation with multiple measurements can be written as [39]

$$\sum_{k=1}^N S(\tilde{\rho}^{(k)}) \geq -\log_2 b + (N-1)S(\rho), \quad (3)$$

where b is a positive quantity no larger than 1 and determined by measurement operators $\mathcal{M}^{(k)}$,

$$b = \max_{i_N} \left[\sum_{i_2, \dots, i_{N-1}} \max_{i_1} [\text{tr}(\Pi_{i_1}^{(1)}, \Pi_{i_2}^{(2)})] \prod_{k=2}^{N-1} \text{tr}(\Pi_{i_k}^{(k)}, \Pi_{i_{k+1}}^{(k+1)}) \right]. \quad (4)$$

$\tilde{\rho}^{(k)}$ is the post-measurement state for $\mathcal{M}^{(k)}$. We remark that quantity b depends only on measurement set $\{\mathcal{M}^{(k)}\}_{k=1}^N$ and no order should be a priori assumed for $\{\mathcal{M}^{(k)}\}_{k=1}^N$ in equation (4). Therefore b is state independent. We adjust (3) to the form

$$\sum_{k=1}^N [S(\tilde{\rho}^{(k)}) - S(\rho)] \geq -\ln b - S(\rho)$$

for the benefit of obtaining complementary relation of quantum coherence. Since relative entropy coherence measure $C_{\text{RE}}^{(k)}(\rho)$ corresponding to reference basis of $\{\Pi_i^{(k)}\}$ was defined as the increase of entropy $S(\rho^{(k)}) - S(\rho)$ of the system due to measurement $\mathcal{M}^{(k)}$, we equivalently can obtain

$$\sum_{k=1}^N C_{\text{RE}}^{(k)}(\rho) \geq -\ln b - S(\rho). \quad (5)$$

This complementary relation provides a lower bound for relative entropy coherence which would be a complementary to the upper bound given in Ref [15]. We can find

that the total coherence as resource should be larger than a bound in different bases. We know that relative entropy of coherence $C_{\text{RE}}^{(k)}(\rho)$ is non-negative, however, the right-hand-side of the inequality can be positive, zero or negative. In particular, when ρ is a mixed state, $S(\rho)$ can be relatively large, so the bound will probably be negative. This fact is reasonable because that coherence depends on the chosen basis, i.e., projective measurement \mathcal{M} . Let us consider an extreme case as an example, when ρ is a completely mixed state, $S(\rho) = \log_2 d$, where d is the dimension. Its coherence is always zero regardless of the basis, so the total coherence in the right-hand-side of (5) is also zero, while right-hand-side, $-\log_2 b - S(\rho) \leq 0$. Then, the inequality is always true for arbitrary sets of $\{\mathcal{M}^{(k)}\}_{k=1}^N$. The equality can be satisfied for some specifically bases such as the mutually unbiased bases. More importantly, this also indicates we can expect higher bound if the state are more purer which is indicated by smaller entropy of the state. We notice that this result (5) was also reported in Ref. [41] very recently.

Complementary relations of thermal discord.— For a composite system consisting of two subsystems A and B, the multi-measurement uncertainty with memory can be derived also [39]

$$\sum_{k=1}^N S(\mathcal{M}_A^{(k)}|B) \geq -\ln b + (N-1)S(A|B). \quad (6)$$

By $S(\mathcal{M}_A^{(k)}|B)$ we mean the conditional information after the measurement $\mathcal{M}_A^{(k)}$ on A. We can rewrite the first term of thermal discord in definition (2) as

$$\begin{aligned} \sum_{i=1}^d p_i(k) S(\tilde{\rho}_{Bi}^{(k)}) &= S\left(\sum_{i=1}^d p_i^{(k)} \Pi_i^{(k)} \otimes \tilde{\rho}_{Bi}^{(k)}\right) - S(\tilde{\rho}_A^{(k)}) \\ &= S(\tilde{\rho}_{AB}^{(k)}) - S(\tilde{\rho}_A^{(k)}). \end{aligned}$$

We recall that $\tilde{\rho}_{Bi}^{(k)}$ is the reduced density operator of subsystem B corresponding to measurement result i after measurement $\mathcal{M}_A^{(k)}$ on subsystem A. $\tilde{\rho}_A^{(k)}$, $\tilde{\rho}_B^{(k)}$ and $\tilde{\rho}_{AB}^{(k)}$ stand for state of subsystems A, B and joint system AB respectively after measurement $\mathcal{M}_A^{(k)}$. Therefore, thermal discord can be expressed in the following manner

$$\begin{aligned} D_{\text{th}}^{(k)}(B|A) &= S(\tilde{\rho}_{AB}^{(k)}) - S(\tilde{\rho}_A^{(k)}) + S(\tilde{\rho}_A^{(k)}) - S(\rho_{AB}) \\ &= [S(\tilde{\rho}_{AB}^{(k)}) - S(\rho_{AB})] - [S(\rho_A) - S(\rho_B)] \\ &= S(\mathcal{M}_A^{(k)}|B) - S(A|B), \end{aligned} \quad (7)$$

where the last line used the fact that $\text{tr}_A \tilde{\rho}_{AB}^{(k)} = \text{tr}_A \rho_{AB} = \rho_B$. It is now straightforward to reexpress (6) as

$$\sum_{k=1}^N D_{\text{th}}^{(k)}(B|A) \geq -\ln b - S(A|B). \quad (8)$$

This serves as a complementary relation for thermal discord with respect to different measurements basis. This lower bound contains a term of conditional information of the pre-measurement state and thus is also state dependent. As we have mentioned the negativity of conditional information signals entanglement, therefore for more entangled state we can expect higher thermal discord which is quite reasonable.

One would notice (8) would be reduced to (5) when dimension of the Hilbert space of B is reduced to 1. This indicates a close relation between relative entropy coherence and thermal discord. In this case, thermal discord would reduce to relative entropy coherence measure and conditional information $S(A|B)$ would equal to $S(\rho_A)$ since $\rho_{AB} = \rho_A$ and $\rho_B = 1$.

Complementary relation of post-measurement conditional information.— In a system of $N + 2$ parties $AB_0 \cdots B_N$, we would consider $N + 1$ measurements $\mathcal{M}_A^{(k)}$ on subsystem A. By introducing an ancillary subsystem C, we can purify $\rho_{AB_0 B_k}$ to $\rho_{AB_0 B_k C}$ which makes $\tilde{\rho}_{B_0 B_k C i}$, the post-measurement state of $B_0 B_k C$ corresponding to result i of measurement $\mathcal{M}_A^{(k)}$ on subsystem A, also a pure state. Then, we can have $S(\rho_{AB_0}) = S(\rho_{B_k C})$ and $S(\tilde{\rho}_{B_0 i}) = S(\tilde{\rho}_{B_k C i})$. Therefore

$$\begin{aligned} & S(\mathcal{M}_A^{(k)}|B_0) - S(A|B_0) \\ &= S\left(\sum_{i=1}^d p_i \Pi_{A i}^{(k)} \otimes \tilde{\rho}_{B_0 i}^{(k)}\right) - S(\rho_{AB_0}) \\ &= S\left(\sum_{i=1}^d p_i \Pi_{A i}^{(k)} \otimes \tilde{\rho}_{B_k C i}^{(k)}\right) - S(\rho_{B_k C}) \\ &\leq S\left(\sum_{i=1}^d p_i \Pi_{A i}^{(k)} \otimes \tilde{\rho}_{B_k i}^{(k)}\right) - S(\rho_{B_k}) \\ &= S(\mathcal{M}_A^{(k)}|B_k), \end{aligned} \quad (9)$$

where in the last but one line of which we have used subadditivity of von Neumann entropy [27] $S(\rho_{AB_k C}) - S(\rho_{B_k C}) \leq S(\rho_{AB_k}) - S(\rho_{B_k})$. Now, equipped with inequality (9) and (6) we can obtain

$$S(A|B) + \sum_{k=1}^N S(\mathcal{M}_A^{(k)}|B_k) \geq -\ln b. \quad (10)$$

Also we know that projective measurements can increase entropy and as a consequence

$$\begin{aligned} S(\mathcal{M}_A^{(k)}|B) - S(A|B) &= S(\tilde{\rho}_{AB}^{(k)}) - S(\rho_{AB}) \\ &\geq 0. \end{aligned}$$

Thus, we can further obtain complementary relation of post-measurements conditional information

$$S(\mathcal{M}_A^{(k)}|B_k)$$

$$\sum_{k=0}^N S(\mathcal{M}_A^{(k)}|B_k) \geq -\ln b, \quad (11)$$

similar to (5) of relative entropy coherence measure and (8) of thermal discord. We note that by B_0 we mean subsystem B. One merit of this bound is that it is state independent. Also it describes constraint on bipartite quantum correlation in a multipartite system (usually with more than two subsystems) which is quite impressing. One can also notice that this inequality is actually a generalization of a similar result in Ref. [40] from two-measurement case to multiple-measurement cas. We believe it can help in the study of quantum correlation concerning multipartite systems.

Conclusion.— By utilizing the uncertainty principle formulated in terms of entropies, we explicitly give a group of complementary relations for quantum coherence, thermal discord and conditional information. Specifically, the entropic uncertainty relation without memory would give us a lower bound (5) on quantum coherence in system of single particle under multiple measurements. It states that the sum of the coherence of quantum state with respect to different measurement basis should not be lower than a bound. Also the purer the state which corresponds to lower entropy, the higher this bound would be. While those entropic uncertainty relation concern additional memories would provide lower bound (8) on thermal discord with respect to different projective measurements on a subsystem of a bipartite joint system. This lower bound is for the sum of thermal discord with respect to multiple measurements on the chosen subsystem and indicates the more entanglement of the state which corresponds to higher negativity of conditional information, the higher this would bound would be. The latter group of complementary relation about thermal discord can be reduced to that of quantum coherence by discard the auxiliary memory through setting its dimension to 1. We also derived a state-independent lower bound for the sum of post-measurement conditional information between a specific subsystem and the other subsystems in a multipartite system with respect to multiple measurements on the first subsystem. These three kinds of complementary relations also indicate a delicate relation between uncertainty principle with quantum coherence and quantum correlation in quantum mechanics.

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